Mathematical Induction

Guess and Prove Formulas by Induction

1. Guess and prove formlas for:

(i) $1 + (1 + 9) + (1 + 9 + 25) + \dots + [1^2 + 3^2 + 5^2 + \dots + (2n - 1)^2]$

(ii) $a + 3(a + b) + 6(a + 2b) + ... + \frac{n}{2}(n + 1)[a + (n - 1)b]$.

2. Guess and prove a law simplifying the product: $2\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\left(1 - \frac{1}{16}\right)...\left(1 - \frac{1}{n^2}\right)$.

3. By calculating $\frac{\sin\left(n + \frac{1}{2}\right)x}{2\sin\frac{1}{2}x} - \frac{\sin\left(n - \frac{1}{2}\right)x}{2\sin\frac{1}{2}x}$, guess and prove a formula for $\frac{1}{2} + \cos x + \cos 2x + ... + \cos nx$.

4. Compute the sum: $S_n = \frac{a}{b} + \frac{a(a-1)}{b(b-1)} + \frac{a(a-1)(a-2)}{b(b-1)(b-2)} + ... + \frac{a(a-1)...(a-n+1)}{b(b-1)...(b-n+1)}$, $b \neq 0, 1, 2, \dots, (n-1)$.

5. Simplify the expression:

$(1 - x)(1 - x^2)\dots(1- x^n) + x(1 - x^2)(1 - x^3)\dots(1- x^n) + x^2 (1 - x^3)\dots(1- x^n) +\dots + x^k (1 - x^{k+1})\dots(1 - x^n) + \dots+ x^{n-1}(1 - x^n) + x^n$.

6. The numbers: $a_0, a_1, a_2, \dots$; $b_0, b_1, b_2, \dots$ are determined by the following law:

$a_{n+1} = \frac{a_n + b_n}{2}$, $b_{n+1} = \frac{2a_n b_n}{a_n + b_n}$, a_0 and b_0 are given, and $a_0 > b_0 .> 0$.

Express a_n and b_n in terms of a_0, b_0 and n.

7. The terms $x_0, x_1, x_2, \dots$ are determined by the equality: $x_n = \frac{\alpha x_{n-1} + \beta}{\gamma x_{n-1} + \delta}$.

Express x_n in terms of x_0 and n.

Consider the particular cases: (i) $x_n = \frac{x_{n-1}}{2x_{n-1} + 1}$, (ii) $x_n = \frac{x_{n-1} + 1}{x_{n-1} + 3}$

8. The terms of the series $x_0, x_1, x_2, \dots$ are connected by the relation $x_n = \frac{px_{n-1} + qx_{n-2}}{p + q}$.

Express x_n in terms of x_0 , x_1 and n.

9. The numbers : $x_0, x_1, x_2, \dots$; $y_0, y_1, y_2 \dots$ are related as follows:

$x_n = \alpha x_{n-1} + \beta y_{n-1}$, $y_n = \gamma x_{n-1} + \delta y_{n-1}$. where $\alpha\delta - \beta\gamma \neq 0$.

Express x_n , y_n in terms of x_0 , y_0 and n.

10. The terms of the series: $x_0, y_0, x_1, y_1, x_2, y_2 \dots$ are determined by the relations:

$x_n = x_{n-1} + 2y_{n-1} \sin^2 \alpha$, $y_n = y_{n-1} + 2x_{n-1} \cos^2 \alpha$.

Besides, it is known that $x_0 = 0$, $y_0 = \cos \alpha$, express x_n and y_n in terms of α.

11. The terms of the series: $a_1, a_2, a_3, \dots$ satisfies the relation: $a_{n+1} - 2a_n + a_{n-1} = 1$, $(n = 2, 3, \dots)$

Express a_n in terms of a_1, a_2 and n.

12. The terms of the series: $a_1, a_2, a_3, \dots$ are related in the following way $a_{n+3} - 3a_{n+2} + 3a_{n+1} - a_n = 1$.

Express a_n in terms of a_1, a_2, a_3 and n.

13. The terms of the series: $a_1, a_2, a_3, \dots$ are connected by the relation: $a_n = ka_{n-1} + l$ $(n = 2, 3, \dots)$

Express a_n in terms of a_1, k, l and n.